SECTION 8.9: IMPROPER INTEGRALS

EXAMPLE 1: Suppose $b > 1$ and let $F(b) = \int_1^b \frac{1}{x^2}\,dx$.

1. What does $F(b)$ represent geometrically?

   Ans: The area between $y = \frac{1}{x^2}$ and the x-axis over the interval $[1, b]$.

2. Find an expression for $F(b)$.

   Ans: $F(b) = 1 - \frac{1}{b}$

3. Find $\lim_{b\to\infty} F(b)$. What does your answer mean geometrically?

   Ans: $\lim_{b\to\infty} F(b) = 1$; this represents ... an area?

DEFINITION: If f is continuous on $[a, \infty)$, then $\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$, provided this limit exists.

If the limit exists, we say that the integral $\int_a^\infty f(x)\,dx$ '**converges**.'

Moreover, if $\int_a^\infty f(x)\,dx = L$, we say that the integral '**converges to** L.'

If the limit does not exist, we say that the integral $\int_a^\infty f(x)\,dx$ **diverges**.

EXAMPLE 2: Rewrite the integrals below using limits to determine which converge and which diverge.

1. $\int_1^\infty \frac{1}{\sqrt{x}}\,dx$

   Ans: $\int_1^\infty \frac{1}{\sqrt{x}}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{\sqrt{x}}\,dx = ... = \infty$; diverges

2. $\int_1^\infty \frac{1}{x}\,dx$

   Ans: $\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx = ... = \infty$; diverges

3. $\int_1^\infty \frac{1}{x^{1.001}}\,dx$

   Ans: $\int_1^\infty \frac{1}{x^{1.001}}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^{1.001}}\,dx = ... = 1000$; converges

THEOREM: The integral $\int_1^\infty \frac{1}{x^p}\,dx$ converges to $\frac{1}{p-1}$ when $p > 1$ and diverges when $p \le 1$.

EXAMPLE 3: We showed earlier that $\int_1^\infty \frac{1}{x}\,dx$ diverges. What about the following integrals?

1. $\int_{117}^\infty \frac{1}{x}\,dx$
2. $\int_{100}^\infty \frac{1}{x}\,dx$
3. $\int_{117^{100}}^\infty \frac{1}{x}\,dx$

Ans: Each of these integrals diverge. It's what happens as $x \to \infty$ which determines convergence or divergence.

MORAL: So long as $a > 0$, the value of 'a' does not affect whether or not the integral $\int_a^\infty \frac{1}{x^p}\,dx$ converges.

NOTE: If $p > 1$ the value of 'a' will affect **the value** of $\int_a^\infty \frac{1}{x^p}\,dx$ - but not the fact that it converges.

EXAMPLE 4: (VIDEO) Rewrite the integral below using limits to determine which converge and which diverge.

1. $\int_0^\infty x^2\,e^{-x}\,dx$

Ans: $\int_0^\infty x^2\,e^{-x}\,dx = \lim_{b\to\infty} \int_0^b x^2\,e^{-x}\,dx = \ldots = 2$; converges

DEFINITION:

- If f is continuous on $(-\infty, a]$, then $\int_{-\infty}^a f(x)\,dx = \lim_{b\to-\infty} \int_b^a f(x)\,dx$, provided this limit exists.
- If f is continuous on $(-\infty, \infty)$ and **both** $\int_{-\infty}^0 f(x)\,dx$ and $\int_0^\infty f(x)\,dx$ converge, then we define:

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^0 f(x)\,dx + \int_0^\infty f(x)\,dx$$

NOTE: If either $\int_{-\infty}^0 f(x)\,dx$ or $\int_0^\infty f(x)\,dx$ diverges, then so does $\int_{-\infty}^\infty f(x)\,dx$.

EXAMPLE 5: (VIDEO) Rewrite the integrals below using limits to determine which converge and which diverge.

1. $\int_{-\infty}^\infty \frac{2}{x^2+4}\,dx$

Ans: $\int_{-\infty}^\infty \frac{2}{x^2+4}\,dx = \int_{-\infty}^0 \frac{2}{x^2+4}\,dx + \int_0^\infty \frac{2}{x^2+4}\,dx = \ldots = \pi$; converges

2. $\int_{-\infty}^\infty x\,e^{-x}\,dx$

Ans: $\int_{-\infty}^0 x\,e^{-x}\,dx$ diverges so $\int_{-\infty}^\infty x\,e^{-x}\,dx$ diverges as well.

EXAMPLE 6: Consider the integral: $\displaystyle\int_0^3 \frac{1}{\sqrt{9-x^2}}\,dx$

1. Why can't we use the Fundamental Theorem of Calculus to evaluate this integral?

   Ans: The integrand, $f(x) = \dfrac{1}{\sqrt{9-x^2}}$, is not continuous on $[0,3]$.

2. Suppose $0 < b < 3$ and let $F(b) = \displaystyle\int_0^b \frac{1}{\sqrt{9-x^2}}\,dx$.

   (a) Find an expression for $F(b)$.

   Ans: $F(b) = \sin^{-1}\left(\dfrac{b}{3}\right)$.

   (b) Find $\displaystyle\lim_{b\to 3^-} F(b)$ and interpret your answer geometrically.

   Ans: $\displaystyle\lim_{b\to 3^-} F(b) = \frac{\pi}{2}$; this represents ... an area?

DEFINITION: If f is continuous on $[a,c)$, then $\displaystyle\int_a^c f(x)\,dx = \lim_{b\to c^-}\int_a^b f(x)\,dx$, provided this limit exists.

If the limit exists, we say that the integral $\displaystyle\int_a^c f(x)\,dx$ '**converges**.'

Moreover, if $\displaystyle\int_a^c f(x)\,dx = L$, we say that the integral '**converges to** L.'

If the limit does not exist, we say that the integral $\displaystyle\int_a^c f(x)\,dx$ **diverges**.

Likewise, if f is continuous on $(a,c]$, then $\displaystyle\int_a^c f(x)\,dx = \lim_{b\to a^+}\int_b^c f(x)\,dx$, provided this limit exists.

EXAMPLE 7: Determine if $\displaystyle\int_{-3}^3 \frac{1}{\sqrt{9-x^2}}\,dx$ converges or diverges.

Ans: $\displaystyle\int_{-3}^3 \frac{1}{\sqrt{9-x^2}}\,dx = \int_{-3}^0 \frac{1}{\sqrt{9-x^2}}\,dx + \int_0^3 \frac{1}{\sqrt{9-x^2}}\,dx = \pi$; converges

EXAMPLE 8: (VIDEO) Rewrite the integrals below using limits to determine which converge and which diverge.

1. $\displaystyle\int_0^2 \frac{x}{\sqrt{4-x^2}}\,dx$

   Ans: $\displaystyle\int_0^2 \frac{x}{\sqrt{4-x^2}}\,dx = \lim_{b\to 2^-}\int_0^b \frac{x}{\sqrt{4-x^2}}\,dx = \ldots = 2$; converges

2. $\displaystyle\int_0^2 \frac{x}{4-x^2}\,dx$

   Ans: $\displaystyle\int_0^2 \frac{x}{4-x^2}\,dx = \lim_{b\to 2^-}\int_0^b \frac{x}{4-x^2}\,dx = \ldots = \infty$; diverges

HOMEWORK: Section 8.9: 7 - 63 odd.